Analysis and Exact Synthesis of Cascaded Commensurate Transmission-Line C-Section All-Pass Networks

EDWARD G. CRISTAL, SENIOR MEMBER, IEEE

Abstract—An analysis of cascaded commensurate transmission-line C-section all-pass networks is presented. The analytical form of the transmission coefficient is found to have a very simple form, intimately related to the reflection coefficient of the stepped-impedance transformer prototype of the cascaded C-section. The phase function of cascaded commensurate transmission-line C-sections is investigated and found to be the arctangent of a reactance function in tan θ. Last, general, exact synthesis procedures for designing cascaded commensurate transmission-line C-section all-pass networks to have prescribed phase characteristics are presented, and two design examples are given. One of the examples is the exact design of a 3-section Schiffman 90° phase shifter, which has not been previously reported in the literature.

Introduction

THE CASCADED commensurate transmission-line C-section all-pass network is shown diagrammatically in Fig. 1. Signals incident to Port 1 are transmitted without reflection to Port 2. The phase lag between Ports 2 and 1, however, may be prescribed by the designer (insofar as the network is capable of realizing the prescribed characteristic). These networks are therefore useful as phase equalizers and phase shifters. It has also been suggested[1] that these networks be used to synthesize a linear time-delay vs. frequency characteristic for use in pulse compression networks. Schiffman [1] used C-sections to construct wide-band 90° differential phase shifters. He presented designs calling for single C-sections, double C-sections, and arrays of single C-sections. Since Schiffman's paper, however, little attention has been given to the general cascaded transmission-line C-section, although recently Shelton and co-workers [2] have described an approximate synthesis procedure based on a first-order theory; and Zysman and Matsumoto [3] have investigated some of the analytic properties of cascaded C-sections.

The present paper presents an investigation of the analytic properties of cascaded commensurate transmission-line C-sections, and gives a general, exact synthesis procedure for realizing cascaded commensurate transmission-line C-sections to have prescribed phase characteristics.

Manuscript received November 26, 1965; revised April 4, 1966.
The author is with the Stanford Research Institute, Menlo Park, Calif.
[1] L. Young, private communication.

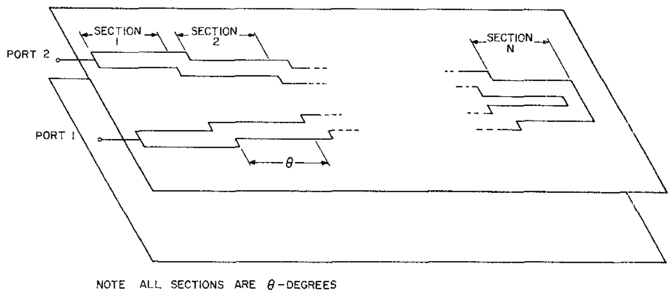

Fig. 1. Cascaded transmission-line C-section all-pass network of N sections.

Analysis

Jones and Bolljohn [4] show that the condition that a single C-section be matched at all frequencies is

$$Z_o = \sqrt{Z_{oe}Z_{oo}}, \tag{1}$$

where Z_o is the characteristic impedance of the coupled lines, Z_{oe} is the even-mode impedance of the coupled lines, and Z_{oo} is the odd-mode impedance of the coupled lines.

Equation (1) is also the condition for a pair of coupled lines to be a directional coupler. Thus, the C-section may be regarded as a directional coupler with two adjacent ports connected by a zero-length line. This analogy carries over directly to cascaded transmission-line C-sections: cascaded transmission-line C-sections may be considered as cascaded transmission-line directional couplers having two adjacent ports connected by a zero-length line.

Let the scattering matrix for a cascaded transmission-line directional coupler be

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 & S_{12} & 0 & S_{14} \\ S_{12} & 0 & S_{14} & 0 \\ 0 & S_{14} & 0 & S_{34} \\ S_{14} & 0 & S_{34} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \tag{2}$$

Consider the effects of connecting Ports 3 and 4 by a zero-length line. Utilizing (2) together with the boundary condition

$$b_4 = a_3, \tag{3}$$

it is easily shown that

$$\frac{b_2}{a_1} = S_{12} + \frac{S_{14}^2}{1 - S_{34}} \cdot \quad (4)$$

Next, using the condition for lossless networks that the scattering matrix S satisfy

$$SS^* = I, \quad (5)$$

where

S^* is the conjugate of S

and

I is a 4×4 identity matrix,

it can be shown that

$$S_{34} = \frac{-S_{12}^* S_{14}}{S_{14}^*} \cdot \quad (6)$$

Substituting (6) into (4) gives

$$b_2/a_1 = \frac{S_{12}S_{14}^* + S_{14}}{S_{12}^* S_{14} + S_{14}^*} \cdot \quad (7)$$

It is clear from (7) that $|b_2/a_1| = 1$, since the right side of the equation is a complex number divided by its conjugate. The phase angle of b_2/a_1 is given by the phase angle of the right side of (7).

At this point it is necessary to introduce the "stepped-impedance transformer-directional coupler" analogy. Feldshtein [5], Young [6], and Levy [7] have shown that every cascaded transmission-line directional coupler has a cascaded stepped-impedance transformer analog, or prototype. They have also pointed out that the reflection coefficient γ of the stepped-impedance transformer prototype and the coupling coefficient S_{12} of the cascaded directional coupler are identical, provided the line impedances of the stepped-impedance transformer are the Z_{oe} of the cascaded directional coupler. Similarly, the transmission coefficient of the stepped-impedance transformer is identical to the transmission coefficient S_{14} of the directional coupler.

Seidel and Rosen [8] have shown that the necessary and sufficient conditions that a function L represent the insertion loss of cascaded commensurate transmission lines is that it be of the form

$$L = \text{Insertion loss} = L_n(\sin^2 \theta), \quad (8)$$

where L_n is a polynomial of degree n; n is the number of cascaded lines, and L is greater than or equal to 1 for all θ. Using Richards' transformation [9]

$$s = j \tan \theta, \quad (9)$$

where $j = \sqrt{-1}$, the insertion loss function may be made a function of s^2. From (9) is obtained

$$\sin^2 \theta = \frac{-s^2}{1 - s^2} \cdot \quad (10)$$

Substituting (10) into (8) gives [13]

$$L = \frac{P_n(s^2)}{(1 - s^2)^n}, \quad (11)$$

where P_n is an nth degree polynomial differing from L_n. Now

$$L = \frac{1}{t_n(s)t_n(-s)} = \frac{1}{1 - \gamma_n(s)\gamma_n(-s)}, \quad (12)$$

where $t_n(s)$ and $\gamma_n(s)$ are, respectively, the transmission and reflection functions of the stepped-impedance transformer prototype. $t_n(s)$ and $\gamma_n(s)$ may be determined from (12) by well-known procedures [14] yielding

$$t_n(s) = \frac{(\sqrt{1 - s^2})^n}{D_n(s)}, \quad (13)$$

and

$$\gamma_n(s) = \frac{N_n(s)}{D_n(s)}, \quad (14)$$

where

$D_n(s)$ is a Hurwitz polynomial[2] chosen from factors of $P_n(s^2) = 0$,

and

$N_n(s)$ is a polynomial (not necessarily Hurwitz) chosen from the factors of $P_n(s^2) - (1 - s^2)^n = 0$.

Recalling the analogies that have been made, we can identify $t_n(s)$ with $S_{14}(s)$, and $\gamma_n(s)$ with $S_{12}(s)$ of (7). Substituting (13) and (14) into (7) gives

$$b_2/a_1 = \frac{N_n(s) + D_n(-s)}{N_n(-s) + D_n(s)} \cdot \quad (15)$$

Equation (15) is a surprisingly simple result which is useful for analysis and, moreover, provides the key to an exact synthesis procedure. The phase shift through the cascaded C-section for real frequencies $s = j \tan \theta$ is given by

$$\beta = 2 \angle N_n(s) + D_n(-s) \big|_{s = j \tan \theta}, \quad (16)$$

where the symbol $\angle$ stands for "the angle of," and β is the phase shift through the cascaded C-section.

The previous results may be summarized briefly as follows. To calculate the phase shift through a cascaded C-section all-pass network, compute the reflection coefficient of the corresponding stepped-impedance transformer prototype using the even-mode impedances of the C-sections for the line impedances of the transformer. Let the reflection coefficient be

$$\gamma_n(s) = \frac{N_n(s)}{D_n(s)} \cdot$$

[2] A Hurwitz polynomial is one whose zeros have negative real parts.

Then the phase shift through the C-section all-pass network is

$$2\angle N_n(s) + D_n(-s)\big|_{s=j\tan\theta}.$$

Synthesis

The synthesis procedure follows readily from (16), (14), (12), and (11). Let $F_n(s)$ be an *admissible* polynomial of degree n defined as

$$N_n(s) + D_n(-s) = F_n(s). \quad (17)$$

The properties of $F_n(s)$ will be described later. For the time being we can say that $F_n(s)$ corresponds to one-half the phase function (i.e., $\beta/2$). Furthermore, it can be shown using (14), (12), and (11) that

$$D_n(s)D_n(-s) - N_n(s)N_n(-s) = (1-s^2)^n. \quad (18)$$

Equations (17) and (18) provide two conditions by by which the polynomials $N_n(s)$ and $D_n(s)$ can be determined.

As a preliminary step, and for future reference, let us make the following definitions:

$$N_n(s) = n_0 + n_1 s + n_2 s^2 + \cdots n_n s^n \quad (19)$$

$$D_n(s) = d_0 + d_1 s + d_2 s^2 + \cdots d_n s^n \quad (20)$$

$$F_n(s) = A_0 + A_1 s + A_2 s^2 + \cdots A_n s^n \quad (21)$$

$$D_e(s) = d_0 + d_2 s^2 + d_4 s^4 + \cdots \quad (22)$$

$$F_e(s) = A_0 + A_2 s^2 + A_4 s^4 + \cdots \quad (23)$$

where the subscript e in (22) and (23) is used to denote the even parts of the polynomials $D_n(s)$ and $F_n(s)$.

$$D_o(s) = d_1 s + d_3 s^3 + d_5 s^5 + \cdots \quad (24)$$

$$F_o(s) = A_1 s + A_3 s^3 + A_5 s^5 + \cdots \quad (25)$$

where the subscript o in (24) and (25) is used to denote the odd parts of the polynomials $D_n(s)$ and $F_n(s)$.

$N_n(s)$ must have a factor of s since $\gamma_n(s)=0$ at $s=0$. Therefore,

$$n_0 = 0. \quad (26)$$

It can also be shown that there is no loss of generality in choosing the constant term of $F_n(s)$ equal to 1. Thus,

$$A_0 = 1, \quad (27)$$

and by (17),

$$d_0 = 1 \quad (28)$$

also. In spite of the simplifications of (26) to (28), we will retain the particular forms of (19) to (21) for notational purposes for equations appearing in the Appendix.

Next, solving (17) for $N_n(s)$ and substituting the result into (18) gives

$$D_n(s)F_n(s) + D_n(-s)F_n(-s) = (1-s^2)^n + F_n(s)F_n(-s). \quad (29)$$

Using the identities

$$D_n(\pm s) = D_e(s) \pm D_o(s) \quad (30)$$

$$F_n(\pm s) = F_e(s) \pm D_o(s), \quad (31)$$

(29) reduces to

$$2[D_e(s)F_e(s) + D_o(s)F_o(s)] = (1-s^2)^n + [F_e(s)]^2 - [F_0(s)]^2. \quad (32)$$

Equation (32) is the fundamental equation that readily permits determination of the coefficients d_i. By substituting (22) to (25) into (32) and equating coefficients of like powers of s, a set of simultaneous *linear* equations is obtained. Solutions of this set give the d_i. Once numerical values for the d_i are known, the coefficients n_i are easily obtained from (16), which reduces to the following set of trivial equations.

$$n_i + (-1)^i d_i = A_i, \quad \text{for } i = 0, 1, \cdots n. \quad (33)$$

It turns out that the above procedure for obtaining the set of linear equations (32) for the d_i can be written down by inspection by using a few simple rules. These rules are given in the Appendix. Also given in the Appendix are explicit solutions for the d_i and n_i for up to 3 sections (i.e., $n=3$) together with the appropriate linear set of equations for the 4-section case.

After the coefficients d_i and n_i have been obtained, the reflection coefficient is constructed according to

$$\gamma_n(s) = \frac{N_n(s)}{D_n(s)}. \quad (14)$$

From $\gamma_n(s)$ the normalized input impedance is determined by the rule

$$Z_n(s) = \frac{1+\gamma_n(s)}{1-\gamma_n(s)}, \quad (34)$$

and last, from $Z_n(s)$ the even-mode impedances of the stopped impedance prototype are extracted [7], [8], [11], [12].

The synthesis procedure may be briefly summarized as follows. To synthesize a prescribed *admissible* phase function $F_n(s)$, the coefficients of the polynomials $N_n(s)$ and $D_n(s)$ are determined by the rules given in the Appendix. Next, the normalized input impedance function is formed using (14) and (34). Last, the even-mode impedances of the stepped-impedance prototype are extracted from the normalized input impedance.

The Phase Function

In order to apply the previously described synthesis procedure, it is necessary to ascertain what are the admissible functions $F_n(s)$. It was previously shown that $\beta/2$ equals the angle of $F_n(s)$ for $s=j\tan\theta$ and, therefore,

$$\beta/2 = \angle F_n(s) = \tan^{-1}\frac{\text{odd part of } F_n(s)}{j \text{ even part of } F_n(s)}\bigg|_{s=j\tan\theta}. \quad (35)$$

Since $\beta/2$ is a monotonic increasing function of θ, it is hypothesized that the argument of the right side of (35) is a *reactance function* in the variable $\tan\theta$. Thus for n even, the general form of the argument of the right side of (35) must be

$$\left.\frac{\text{odd part of } F_n(s)}{j[\text{even part of } F_n(s)]}\right|_{s=j\tan\theta} = \frac{-B_1\tan\theta(1-B_3{}^2\tan^2\theta)\cdots}{(1-B_2{}^2\tan^2\theta)\cdots(1-B_n{}^2\tan^2\theta)}, \quad (36)$$

and for n odd

$$\left.\frac{\text{odd part of } F_n(s)}{j[\text{even part of } F_n(s)]}\right|_{s=j\tan\theta} = \frac{-B_1\tan\theta(1-B_3{}^2\tan^2\theta)\cdots(1-B_n{}^2\tan^2\theta)}{(1-B_2{}^2\tan^2\theta)\cdots(1-B_{n-1}{}^2\tan^2\theta)}, \quad (37)$$

where the coefficients B_i are positive real constants which satisfy

$$B_2 > B_3 > B_4 \cdots > B_n \quad (38)$$

and[3]

$$B_1 \leq n. \quad (39)$$

Once the constants B_i are ascertained, $F_n(j\tan\theta)$ can be constructed by adding the denominator to j times the numerator. $F_n(s)$ is obtained by replacing $j\tan\theta$ by s and $-\tan^2\theta$ by s^2.

Equations (36) and (37) display clearly the characteristic of the phase functions of cascaded commensurate transmission-line C-section all-pass networks. Furthermore, the characteristics of reactance functions are described in detail in the literature [10]. It is seen from (36) and (37) that the designer is free to choose the values of θ for which $\beta/2$ equals 90°, 180°, 270°, etc., by choosing the locations of the poles and zeros. The values of θ for which $\beta/2$ equals 90°, 270°, 450°, etc., correspond to the zeros of the denominator. The values of θ for which $\beta/2$ equals 180°, 360°, 540°, etc., correspond to zeros of the numerator. The remaining variable, B_1, leaves to the designer one parameter to "shape" the phase function or impose one other condition. Once the reactance function is defined, the constants A_i of $F_n(s)$ are easily obtained and the previously described synthesis procedure may be applied.

Examples

Two examples are presented next to order to illustrate the foregoing theory. In the first example, a Schiffman 90° differential phase shifter of 1 C-section [1] is "redesigned." From (37) for $n=1$,

$$\beta/2 = \tan^{-1}(-B_1\tan\theta). \quad (40)$$

A 90° differential phase shifter is obtained by using a reference line which is 3θ long so that, at $\theta=90°$, the differential phase shift is automatically 90°. This is seen from the equation,

$$-3\theta - 2\tan^{-1}(-B_1\tan\theta) = -90° \text{ at } \theta = 90°. \quad (41)$$

Suppose it is decided to make the slope of the phase of the reference line and the C-section equal at $\theta=90°$. Then

$$\frac{d(-3\theta)}{d\theta} = -3 = \frac{d}{d\theta}2\tan^{-1}\{-B_1\tan\theta\} = \left.\frac{-2B_1\sec^2\theta}{1+B_1{}^2\tan^2\theta}\right|_{\theta=90°} = \frac{-2}{B_1}. \quad (42)$$

Thus,

$$B_1 = \tfrac{2}{3}, \quad (43)$$

and

$$\left.\begin{aligned} F_1(j\tan\theta) &= 1 - j\,2/3\tan\theta \\ F_1(s) &= 1 - 2/3\,s \end{aligned}\right\}. \quad (44)$$

From (69) and (70) in the Appendix, it is determined that

$$\left.\begin{aligned} d_1 &= \frac{13}{12}, \\ n_1 &= 5/12. \end{aligned}\right\} \quad (45)$$

Therefore,

$$\gamma_1(s) = \frac{n_1(s)}{d_0 + d_1 s} = \frac{5/12\,s}{1+\frac{13}{12}s}. \quad (46)$$

The input impedance of the transformer prototype is

$$Z_{in} = \frac{1+\gamma_1(s)}{1-\gamma_1(s)} = \frac{1+3/2\,s}{1+2/3\,s}, \quad (47)$$

from which Z_{oe}/Z_o is determined to be

$$\frac{Z_{oe}}{Z_o} = 3/2.$$

The reciprocal of Z_{oe}/Z_o is Z_{oo}/Z_o so that [4]

$$\frac{Z_{oo}}{Z_0} = 2/3.$$

The ratio of Z_{oe} to Z_{oo} is 2.25 which is near the value given in Schiffman's paper [1] for the same design. The two results, of course, should not agree precisely since Schiffman chose other criteria than equal slopes at $\theta=90°$.

The second example will be the synthesis of a 3-

[3] The restriction $B_1 \leq n$ can be justified by noting that the slope of the phase function at $\theta=0$ is $-2B_1$. Since the absolute value of the slope at $\theta=0$ must be less than the slope of uncoupled lines of length $2n\theta$, we have

$$2B_1 \leq 2n$$
$$B_1 \leq n.$$

section C-section 90° differential phase shifter. The appropriate phase function is, by (37),

$$\beta/2 = \tan^{-1}\left\{\frac{-B_1 \tan\theta(1 - B_3^2\tan^2\theta)}{(1 - B_2^2\tan^2\theta)}\right\}. \quad (48)$$

The reference line is chosen 7θ long so that the differential phase shift is given by

$$\Delta\Phi = 2\left[-3.5\,\theta - \tan^{-1}\frac{-B_1\tan\theta(1 - B_3^2\tan^2\theta)}{(1 - B_2^2\tan^2\theta)}\right]. \quad (49)$$

The differential phase shift is automatically $-90°$ at $\theta = 90°$ by virtue of having selected the reference line to be 7θ long. We may also choose B_2 and B_3 to give exactly 90° phase shift at two other frequencies.

At $\beta/2 = -90°$,

$$1 - B_2^2\tan^2\theta\Big|_{\theta=\theta_2} = 0. \quad (50)$$

Therefore, at $\theta = \theta_2$, (49) reduces to

$$\Delta\Phi = 2\{-3.5\theta_2 + 90°\}. \quad (51)$$

Letting $\Delta\Phi = -90°$ and solving (51) gives

$$\theta_2 = \frac{270}{7} = 38.57°. \quad (52)$$

Substituting (52) into (50) gives

$$B_2^2 = \cot^2 38.57° = 1.57256. \quad (53)$$

In a similar way, it is found that, at $\theta = \theta_3$,

$$7\theta_3 - 360° = 90°, \quad (54)$$

$$\theta_3 = 64.28°, \quad (55)$$

and

$$B_3^2 = \cot^2 64.28° = 0.23202. \quad (56)$$

The coefficient B_1 is chosen to limit the maximum deviation from 90° to a specified amount over as wide a frequency band as possible.

To determine B_1, it was assumed that the phase characteristic would have a maximum deviation from 90° at a value of θ roughly halfway between 38.5° and 64.3°. Several trial values of B_1 were substituted into (49) and $\Delta\Phi$ was calculated. It was quickly found that for $\theta = 50°$ and $B_1 = 1.8$, $\Delta\Phi = -88.6°$. Therefore, this value of B_1 was selected. Next, (49) was computed for $5° \leq \theta \leq 90°$ giving the result shown in Fig. 2. It was determined from the data of Fig. 2 that the differential phase shift was $-90° \pm 4°$ over a 4.52:1 frequency band.[4] Having determined a value for B_1, the 3-section phase shifter was synthesized as follows:[5] Using (53), (56), and a value of 1.8 for B_1, the function $F_3(s)$ was determined to be

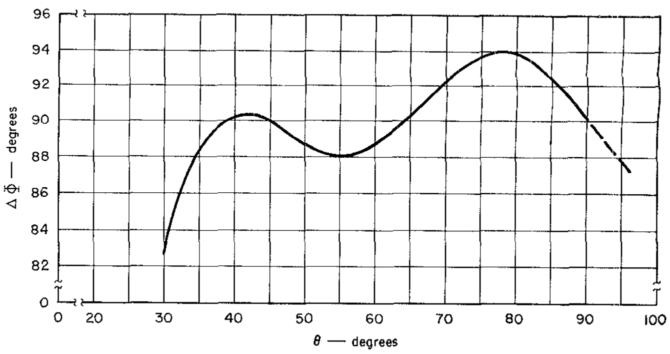

Fig. 2. Differential phase shift of 3-section 90° phase shifter.

$$F_3(s) = 1 - 1.8s + 1.57256s^2 - 0.41763s^3. \quad (57)$$

Next, from (75) through (80) in the Appendix, it was found that

$$\begin{aligned} d_3 &= 1.40604 & n_3 &= 0.98842 \\ d_2 &= 3.90848 & n_2 &= -2.33592 \\ d_1 &= 3.90471 & n_1 &= 2.10471 \\ d_0 &= 1. \end{aligned} \quad (58)$$

Therefore,

$$\gamma_3(s) = \frac{N_3(s)}{D_3(s)} = \frac{2.10471s - 2.33592s^2 + 0.98842s^3}{1 + 3.90471s + 3.90848s^2 + 1.40604s^3}. \quad (59)$$

From $\gamma_3(s)$, $Z_{in}(s)$ was calculated giving

$$Z_{in}(s) = \frac{1 + 6.00942s + 1.57256s^2 + 2.39446s^3}{1 + 1.8s + 6.2444s^2 + .41762s^3}. \quad (60)$$

The normalized even-mode impedances $(Z_{oe}/Z_o)i$ were next extracted using the techniques given in [12]. The results are:

$$\left(\frac{Z_{oe}}{Z_o}\right)_1 = 1.16005 \quad (61)$$

$$\left(\frac{Z_{oe}}{Z_o}\right)_2 = 1.58264 \quad (62)$$

and

$$\left(\frac{Z_{oe}}{Z_o}\right)_3 = 3.26829. \quad (63)$$

In order to check these results, these values of normalized even-mode impedances were substituted into expressions which gave the phase shift β directly in terms of the even-mode impedances. Several values of θ were tried and the calculated phase shift agreed with the values predicted by the theory.[6]

[4] Based on the results given in Fig. 2, it is probable that a value of B_1 of about 1.85 or 1.9 would be closer to an optimum value.

[5] The synthesis which follows was carried out with a desk calculator carrying only 5 digits after the decimal point. Hence, some roundoff error is anticipated in the result.

[6] The expressions for β in terms of the normalized even-mode impedances were derived by B. M. Schiffman, who also kindly checked the results for several values of θ.

Discussion

This paper has considered three aspects of cascaded commensurate transmission-line C-section all-pass networks:

1) analysis
2) a theoretical solution for exact synthesis of prescribed, admissible, phase shift functions, and
3) a description of the basic form of the phase shift function in the variable $\tan\theta$.

A fourth aspect, important but clearly distinct from Items 1, 2, and 3 above, is the approximation problem. Certainly, in any given practical problem it is unlikely that the designer will be given a specified phase function that fits the form of (36) or (37). Therefore, in most instances it will be necessary to obtain an approximation to the required function. Often an equal-ripple approximation is desired: the optimum approximation. This aspect of the synthesis problem has not been considered in this paper. However, it is important to emphasize that the form of the phase function exhibited by (36) and (37), while not making readily evident an optimum solution to the approximation problem, does permit considerable insight into a practical solution to the approximation problem. It is clear from previous discussion and from (36) and (37) that when the required phase function is an integer multiple of 90°, the right side of (36) or (37) must be in the vicinity of a pole or zero. As exemplified in the design of the 3-section 90° phase shifter, it is likely that suitable choices of values for B_i in most applications are those which give zeros or poles at precisely the prescribed points. This then leaves a single parameter to optimize the response. In addition, it is evident that the values of B_i $(i\geq 2)$ cannot vary much from these values. In this respect, one can conjecture that, at least in the general case, equal-ripple approximations to prescribed phase shift function (even linear phase shift functions) are not always obtainable. The reason for this is that the coefficients B_i $(i\geq 2)$ are not variable over the entire range of real numbers but only over an extremely restricted range.

The results obtained from the 3-section 90° phase shifter indicate that the Z_{oe} tend to increase in value in the interior parts of cascaded C-sections. This is analogous to what happens in cascaded directional couplers. The coupling in decibels required by the 3rd section is [12]

$$-20\log_{10}\frac{(Z_{oe}/Z_o)^2-1}{(Z_{oe}/Z_o)^2+1}\ \text{dB}, \tag{64}$$

which gives a value of 1.63 dB. This requires very tightly coupled lines, indeed. It can be anticipated that for larger numbers of cascaded C-sections, the required coupling in the interior sections will be even greater, and therefore impractical to realize with the present state of the art. One method [2] that may be used to get around this is to use several groups of cascaded C-sections in tandem, wherein each group provides only a specified fraction of the desired total phase shift. Thus, for example, if a 6-section C-section 90° differential phase shifter is called for, it might be better to design two 3-section C-section 45° phase shifters and connect them in tandem.

Conclusions

Cascaded transmission-line C-section all-pass networks are needed in many microwave systems that require phase shaping or phase equalization. The formulas for analysis of commensurate transmission-line C-sections all-pass networks are relatively easy to use and should prove useful for diagnostics. The synthesis method that was presented permits exact design of cascaded commensurate transmission-line C-section all-pass networks from admissible prescribed phase functions.

It was also found that the phase characteristic of cascaded commensurate transmission-line C-sections is the arctangent of a reactance function in the variable $\tan\theta$. This result should enable the designer to determine more easily what phase characteristic can be realized or approximated, as well as providing insight into the approximation problem. The formulas which were presented for designs of 1-, 2-, and 3-section C-sections require only a slide-rule or desk calculator. However, for a wide range of synthesis problems it will probably be advantageous to use a digital computer. More than 3 sections will almost certainly require a computer.

The example design of a 3-section 90° differential phase shifter had a tolerance of $\pm 4°$ over a 4.52:1 frequency band. The tolerance could probably be improved by choosing the design constant B_1 equal to 1.85 or 1.9.

Appendix

General Formulas for $N_n(s)$ and $D_n(s)$ for Cascaded Commensurate-Transmission Line C-Sections

$$\gamma_n(s) = N_n(s)/D_n(s) \tag{65}$$

$$N_n(s) = n_1 s + n_2 s^2 + n_3 s^3 + \cdots n_n s^n \tag{66}$$

$$D_n(s) = d_0 + d_1 s + d_2 s^2 + \cdots d_n s^n \tag{67}$$

$$F_n(s) = A_0 + A_1 s + A_2 s^2 + \cdots A_n s^n, \tag{68}$$

where in (67) and (68) A_0 and d_0 are equal to 1.

One Section

$$d_1 = (-A_1^2 - 1)/(2A_1) \tag{69}$$

$$n_1 = (A_1^2 - 1)/(2A_1). \tag{70}$$

Two Sections

$$d_2 = (A_2^2 + 1)/(2A_2) \tag{71}$$

$$n_2 = (A_2^2 - 1)/(2A_2) \tag{72}$$

$$d_1 = -(2 + 2d_2 + A_1^2)/(2A_1) \tag{73}$$

$$n_1 = A_1 + d_1. \tag{74}$$

Three Sections

$$d_3 = (-A_3^2 - 1)/(2A_3) \tag{75}$$

$$n_3 = (A_3^2 - 1)/(2A_3) \tag{76}$$

$$d_2 = \frac{3(n_3 - d_3) - A_1^2(d_3 + n_3) + A_1(3 + A_2^2)}{2(d_3 + A_1A_2 - n_3)} \tag{77}$$

$$n_2 = A_2 - d_2 \tag{78}$$

$$d_1 = -(3 + A_1^2 + 2d_2)/(2A_1) \tag{79}$$

$$n_1 = A_1 + d_1. \tag{80}$$

Four Sections

The coefficients d_i are given by the solution to the following system of equations:

$$\begin{bmatrix} 2A_0 & 0 & 0 & 0 & 0 \\ 2A_2 & 2A_1 & 2A_0 & 0 & 0 \\ 2A_4 & 2A_3 & 2A_2 & 2A_1 & 2A_0 \\ 0 & 0 & 2A_4 & 2A_3 & 2A_2 \\ 0 & 0 & 0 & 0 & 2A_4 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 + 2A_2 - A_1^2 \\ 6 + 2A_4 + A_2^2 - 2A_1A_3 \\ -4 + 2A_2A_4 - A_3^2 \\ 1 + A_4^2 \end{bmatrix}. \tag{81}$$

The coefficients n_i are obtained using the d_i from above and (34) of the text.

General Case of n Sections

In the general case the coefficients d_i are obtained from the solution of the following linear set of equations.

$$\sum_{j=0}^{n} a_{ij}d_j = C_i, \quad \text{for } i = 0, 1, \cdots n. \tag{82}$$

For the following equations define

$$A_p = 0 \quad \text{for } p < 0 \text{ or } > n, \tag{83}$$

where p is a dummy index.

The coefficients a_{ij} are given by

$$a_{ij} = 2A_{2i-j} \quad \text{for } i, j = 0, 1, \cdots n. \tag{84}$$

The C_i are given by

$$C_i = b_i + (f_e)_i - (f_o)_i, \quad \text{for } i = 0, 1, \cdots n \tag{85}$$

where

$$b_i = \frac{(-1)^i n!}{i!(n-i)!} \tag{86}$$

$$(f_e)_i = \sum_{j=0,2} A_jA_{2i-j}, \quad (j \text{ even and } \leq n) \tag{87}$$

$$(f_o)_i = \sum_{j=1,3} A_jA_{2i-j}, \quad (j \text{ odd and } \leq n). \tag{88}$$

The coefficients n_i are obtained from (34) after the coefficients d_i have been numerically evaluated.

Note that the forms of the rules given in (84) through (88) are such as to make it possible to write down (82) in matrix form by inspection. Let the first row and column of the matrix be initialized as the zero row and zero column. Then the a_{ij} entries in the coefficient matrix have the property that the sum of the jth column and the subscript of A equals $2i$. If this is not possible, the entry is 0. The column matrix of constants, C_i, consists of 3 terms: b_i, $(f_e)_i$, and $(f_o)_i$. The constants b_i are the binomial coefficients multiplied by $(-1)^i$. The constants $(f_e)_i$ are formed by summing all possible terms of the form A_mA_n, where m and n are even integers (0 is taken as even) satisfying $m+n=2i$. When $m \neq n$ a particular combination will occur twice, once for A_mA_n and the other A_nA_m. When $m=n$, the combination appears only once. Combinations formed such that $m+n \neq 2i$ are taken as 0. An analogous rule holds for the constants $(f_o)_i$, except that m and n are odd integers.

References

[1] B. M. Schiffman, "A new class of broad-band microwave 90 degree phase shifters," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-6, pp. 232–237, April 1958.

[2] J. P. Shelton, R. Van Wagner, and J. J. Wolfe, "Tandem couplers and phase shifters for multi-octave bandwidth," *Microwaves*, pp. 14–20, April 1965.

[3] G. I. Zysman and A. Matsumoto, "Properties of microwave C-sections," *IEEE Trans. on Circuit Theory*, vol. CT-12, pp. 74–82, March 1965.

[4] E. M. T. Jones and J. T. Bolljahn, "Coupled-strip-transmission-line filters and directional couplers," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-4, pp. 75–81, April 1956.

[5] A. L. Feldshtein, "Synthesis of stepped directional couplers," *Radiotekn. i Electron.*, vol. 6, pp. 234–240, February 1961. (English translation, New York: Pergamon, pp. 74–85.)

[6] L. Young, "The analytical equivalence of TEM-mode directional couplers and transmission-line stepped-impedance filters," *Proc. IEE (London)*, vol. 110, pp. 275–281, February 1963.

[7] R. Levy, "General synthesis of asymmetric multi-element coupled-transmission-line directional couplers," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-11, pp. 226–237, July 1963.

[8] H. Seidel and J. Rosen, "Multiplicity in cascade transmission line synthesis," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-13, pt. I, pp. 275–283, May 1965, and pt. II, pp. 398–407, July 1965.

[9] P. I. Richards, "Resistor-transmission-line circuits," *Proc. IRE*, vol. 36, pp. 217–220, February 1948.

[10] E. A. Guillemin, *Communication Networks*, vol. II. New York: Wiley, 1935.

[11] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-5, pp. 36–43, January 1957.

[12] E. G. Cristal and L. Young, "Theory and tables of optimum symmetrical TEM-mode coupled-transmission-line directional couplers," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-13, pp. 544–559, September 1965.

[13] H. J. Carlin, "Network synthesis with transmission line elements," Rome Air Development Center, Griffiss Air Force Base, N. Y., Tech. Rept. RADC-TDR-64-505, February 1965.

[14] E. A. Guillemin, *Synthesis of Passive Networks*. New York: Wiley, 1957, ch. 11.